Stopping Times

Context : A continuous Filtration :

$$(\Omega, \mathcal{F}_\infty, \mathbb{P}, (\mathcal{F}_t), t \geq 0).$$

A random variable $\tau : \Omega \to [0,\infty]$ is called a stopping time iff $\forall t$,

$$\{\tau \leq t\} \in \mathcal{F}_t .$$

The idea is that at time t we should know whether or not "τ has occured". This is expressed mathematically as $\{\tau \leq t\} \in \mathcal{F}_t$, i.e. that τ has occured before, or at, time t is part of the current information.

There are a lot of stopping times! Let $t_0 \in [0,\infty]$. The random variable $\tilde{t}_0 = t_0 I_\Omega$ is a stopping time.

Exercise : Prove it!

So every deterministic time, t_0, can be identified with a stopping time. There are a lot more stopping times however;

Let $E \in \mathcal{F}_1$. Set $\tau = I_E + 2 I_{\Omega \setminus E}$.

Then $\tau = 1$ on E and 2 on $\Omega \setminus E$. This τ is a stopping time.

Exercise: Prove it.

Imagine how many times one can generate in this way by allowing the values of τ to vary and E to vary over $\mathcal{F}_t$, $t \geq 0$.

What are stopping times for?

(I) Analysis: If $X = (X_t)$ is a process you can 'look at' the process at a random time τ:

$$X_\tau \triangleq X_{\tau(\omega)}(\omega)$$

We call X_τ, "the process X stopped at time τ". If you "know" X_τ at time τ for <u>every</u> stopping time τ, then effectively you have an analysis of X because you know where it has been and when it was there.

II Applications: Suppose you have an asset, S, whose value at time t is given by

$$S_t = S_0 e^{\sigma W_t + (r - \sigma^2/2)t}.$$

One can write options on S — which you have already encountered — with a payoff of $(S_T - K)^+$ at time T. Sometimes these option contracts have an extra condition: If, during $[0, T]$, S_t achieves the value $H > S_0$, then the option becomes worthless. It is said to "knock out". Contracts like this are called Barrier Options. Accordingly we would have an interest in

$$\tau(\omega) = \inf\{t : S_t(\omega) \geq H\} \equiv \text{First time } S \text{ is } \equiv \text{equal to } H$$

We would at least be interested in $\mathbb{P}\{\tau > T\}$ since this is the probability that our Option will not knock out.

III We suppose that we have sold an Option to Mrs Brown and we stated to her that she may exercise it at <u>any</u> time in $[0, T]$ with the payoff $(S_t - K)^+$ $t \in [0, T]$. What we missed when we did this is twofold;

(i) Mrs Brown is very smart and is not to be confused with her husband ..... accordingly.....

(ii)...She can choose exercise at at any stopping time τ.

So, effectively, we have

sold Mrs B one option contract for every stopping time, τ, and we have to find a way of hedging all of these liabilities because we don't know which she may choose..... *

* In fact the mathematical model restricts the behaviour of the parties involved. One is not allowed to be a source of arbitrage.... .